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# MAXIMUM STABILITY FORMULAS FOR REINFORCED CYLINDRICAL SHELLS UNDER EXTERNAL PRESSURE

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The problem of determining the reinforcement structure that maximizes the stability of a cylindrical shell subjected to external pressure was formulated in [1], where a numerical solution was obtained for a particular class of structures on the basis of a formula for the stability limit of a hinged anisotropic circular cylindrical shell of medium length in the membrane state. In the present article the stability limit is determined more accurately, without any constraint on the length of the shell, and the optimization is carried out over a broader class of structures.

§1. Let us consider a circular cylindrical shell of constant thickness H, mean radius R, and length L, made of fibrous composite material. It is assumed that the material has a regular layered structure, so that it is possible to distinguish a typical layer whose thickness is small as compared with that of the shell; the typical layer has multidirectional reinforcement symmetrical with respect to an arbitrary axial section of the shell; the fibers in all directions are made of the same linear-elastic material; the matrix material is linear-elastic and isotropic.

In order to describe the state of stress and strain of the typical layer, we will employ the mechanical model proposed in [2]. Under the assumptions formulated above, this model substitutes for an element of the reinforced layer the statically equivalent element of an orthotropic-elastic homogeneous layer whose state of stress and strain is determined in the principal surface coordinate system by the symmetric plane tensors of the average stresses $\sigma_{ij}$ and strains $\varepsilon_{ij}$ (the subscripts i, j run through the values 1, 2). The equations of [2] for the relationship between the components of these tensors, simplified in accordance with the starting assumptions, take the form

$$\sigma_{11} = \omega E(a_{11}\varepsilon_{11} + a_{12}\varepsilon_{12}),\ \sigma_{12} = \omega E a_{33}\varepsilon_{12},\ 1 \rightleftarrows 2; \tag{1.1}$$

$$a_{11} = \varepsilon + \sum_{k=1}^{K} \frac{\omega_k}{\omega}\chi_{1k}^2, \quad a_{12} = \varepsilon\nu_0 + \sum_{k=1}^{K}\frac{\omega_k}{\omega}\chi_{1k}^2\chi_{2k}^2, \quad 1 \rightleftarrows 2, \tag{1.2}$$

$$a_{33} = \varepsilon(1-\nu_0) + 2\sum_{k=1}^{K}\frac{\omega_k}{\omega}\chi_{1k}^2\chi_{2k}^2, \quad \omega_k \geqslant 0, \quad \omega = \sum_{k=1}^{K}\omega_k < 1,$$

$$\varepsilon = (1-\omega)E_0/(1-\nu_0^2)E\omega > 0, \quad 0 \leqslant \nu_0 \leqslant 1/2,$$

where $E_0$ and E are the moduli of elasticity of the matrix and the fibers, respectively; $\nu_0$ is the Poisson ratio of the matrix; $\omega_k$(k = 1, 2, ..., K) is the volume fraction of fibers of direction k (K is the total number of directions); $\omega$ is the volume fraction of reinforcement; $\chi_{ik}$ are the direction cosines of the k-th direction with respect to the i-th coordinate line.

For shell strains satisfying the Kirchhoff kinematic hypotheses we can write

$$\varepsilon_{ij} = p_{ij} + \zeta q_{ij},$$

where $p_{ij}$, $q_{ij}$ are the symmetric tensors of the tangential and bending strains of the middle

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surface; $\zeta$ is the coordinate normal to the middle surface. Let $N_{ij}$ be the symmetric tangential force tensor and $M_{ij}$ the symmetric internal moment tensor of the shell. By means of the usual procedure of integration over the normal coordinate, from (1.1) we establish the following relations between the shell tensors:

$$M_{11} = D(a_{11}q_{11} + a_{12}q_{22}),\ M_{12} = Da_{33}q_{12}, \tag{1.3}$$

$$p_{22} = B(b_{12}N_{11} + b_{11}N_{22}),\ p_{21} = Bb_{33}N_{21},\ 1 \rightleftarrows 2;$$

$$b_{11} = a_{11}/a,\ b_{22} = a_{22}/a,\ b_{12} = -a_{12}/a,\ b_{33} = 1/a_{33},$$

$$a = a_{11}a_{22} - a_{12}^2, \quad B = 1/HE\omega, \quad D = H^3E\omega/12. \tag{1.4}$$

The equations obtained for the orthotropic-elastic layer are used to determine the stability limit of a shell with hinged ends subjected to uniform lateral pressure. Assuming that it the shell is sufficiently thin, we neglect the effect of the precritical bending strain on the stability limit. In order to investigate the stability of the membrane state of the shell, we use the equations of nonshallow shells represented in mixed (staticogeometric) form [3] (the resolvent system of two equations is obtained for the orthotropic in the same way as for the isotropic shell). These equations give the following stability conditions for the membrane state (p is the lateral pressure intensity, m = 1, 2, 3, ... is the number of half-waves along the generator, n = 2, 3, 4, ... is the number of waves along the circumference):

$$q < q^* = \min_{m,n} Q; \tag{1.5}$$

$$Q = tn^2Q_1 + r^4m^4n^{-6}t^{-3}/Q_2, \tag{1.6}$$

$$Q_1 = a_{22}(1 - n^{-2}) + 2[(1 - n^{-2})a_{12} + a_{33}]r^2n^{-2}m^2 + a_{11}r^4n^{-4}m^4,$$

$$Q_2 = b_{22}(1 - n^{-2}) + 2[(1 - n^{-2})b_{12} + b_{33}]r^2n^{-2}m^2 + b_{11}r^4n^{-4}m^4,$$

$$q = p/(\sqrt{12}t^5\omega E),\ r = \pi R/L,\ t^2 = H/\sqrt{12}R.$$

The critical value $q^*$ of the load parameter q determines the stability limit of the shell under lateral pressure. Since, in accordance with (1.2), the elastic coefficients $a_{ij}$, $a_{33}$, $b_{ij}$, $b_{33}$ entering into expression (1.5) depend on the structure of the composite and on the elastic properties of its individual components, there is a real possibility of controlling the stability limit of the shell.

§2. We will formulate the following optimization problem [1]: For given elastic characteristics of the reinforcement and matrix materials and constant volume reinforcement fraction, determine the reinforcement structure corresponding to the maximum critical load.

First we isolate the independent optimization parameters and investigate their regions of possible values. Let

$$\varphi = \sum_{k=1}^{K} \frac{\omega_k}{\omega} \chi_{1k}^2, \quad \psi = \sum_{k=1}^{K} \frac{\omega_k}{\omega} \chi_{1k}^2\chi_{2k}^2. \tag{2.1}$$

Then expressions (1.2) take the form

$$a_{11} = \varepsilon + \varphi - \psi,\ a_{22} = 1 + \varepsilon - \varphi - \psi,$$

$$a_{12} = \psi + \nu_0\varepsilon,\ a_{33} = 2\psi + (1 - \nu_0)\varepsilon, \tag{2.2}$$

which expresses the dependence of the elastic coefficients on the two parameters $\varphi$, $\psi$ (the constants $\varepsilon$ and $\nu_0$ are given). The specific reinforcement structure corresponding to fixed values of the parameters $\varphi$ and $\psi$ is determined from Eqs. (2.1) nonuniquely, generally speaking. Accordingly, the optimum value of these parameters may correspond to a whole series of optimum reinforcement structures.

The region of variation of the parameters $\varphi$, $\psi$ cannot be arbitrary since they are derived from the parameters $\omega_k$, $\chi_k$, which are subject to the conditions

$$\omega_k \geqslant 0, \quad \sum_{k=1}^{K} \omega_k/\omega = 1, \quad 0 \leqslant \chi_{1k}^2 \leqslant 1, \quad \chi_{2k}^2 = 1 - \chi_{1k}^2,$$

using which it is possible to show that all the values of $\varphi$ and $\psi$ corresponding to some reinforcement structure lie in the region

$$\Omega : 0 \leqslant \varphi \leqslant 1,\ 0 \leqslant \psi \leqslant \varphi(1 - \varphi).$$

In fact, the first constraint follows directly from (2.1), since the quantities $\chi_{1k}^2$ (k = 1, 2, ..., K) are independent. Then let $J \leq K$ be a number such that

$$\chi_{1k}^2\chi_{2k}^2 \leqslant \chi_{1J}^2\chi_{2J}^2 = \chi_{1J}^2(1-\chi_{1J}^2).$$

Here the equal sign is reached when $\chi_{ij} = \chi_1$ (k = 1, 2, ..., K), i.e., when, in accordance with (2.1), $\chi_{1J}^2 = \varphi$. This means that

$$\chi_{1k}^2\chi_{2k}^2 \leqslant \varphi(1-\varphi).$$

Applying this inequality to expression (2.1) for $\psi$, we establish the second of the constraints on $\Omega$.

In the coordinate system $\varphi$, $\psi$ the region $\Omega$ is the segment of the parabola $\psi = \varphi(1-\varphi)$, cut off by the straight line $\psi = 0$. Each point on the straight boundary of the region $\Omega$ defines two families of fibers, one directed along the circumference, the other along the generator, and each point on the parabolic boundary defines two symmetric oblique families. Exceptions are formed by the two corner points $\varphi = 0$, $\psi = 0$ and $\varphi = 1$, $\psi = 0$. The first corresponds to a circumferential family only, the second to an axial family only.

The optimization problem consists in finding those values of the orthotropy parameters $\varphi$, $\psi$ which give the greatest value of the critical load parameter

$$q^+ = \max_{\varphi,\psi} q^* = \max_{\varphi,\psi}\min_{m,n} Q. \tag{2.1}$$

If the values of $\varphi$, $\psi$ from $\Omega$ are such that the inequality

$$b_{12} + b_{33} \geqslant 0 \tag{2.4}$$

is satisfied, then the value of the parameter m realizing min Q is equal to unity for any $n \geq 2$, since both terms in expression (1.6) are functions increasing with respect to m. In fact, from (2.2) and (1.4) there follow the estimates, valid in the region $\Omega$,

$$a_{11} \geqslant \varepsilon + \varphi^2 > 0,\ a_{22} \geqslant \varepsilon + (1-\varphi)^2 > 0,\ 0 < a_{12} \leqslant \nu_0\varepsilon + \varphi(1-\varphi),$$
$$a \geqslant (1-\nu_0^2)\varepsilon^2 + [(1-2\varphi)^2 + 2\varphi(1-\varphi)(1-\nu_0)]\varepsilon > 0,\quad b_{11} > 0,\ b_{22} > 0,$$

which, together with condition (2.4), ensure (at $n \geq 2$) the positiveness of all the coefficients of the expressions for $Q_1$ and $Q_2$ considered as polynomials in m. Accordingly, as m increases, the function $Q_1$ increases, the function $Q_2/m^4$ decreases, and the reciprocal function $m^4/Q_2$ increases.

We note that condition (2.4) may be violated on the parabolic boundary of the region $\Omega$, where, in fact, the parameters $a_{ij}$, $a_{33}$ take values comparable with unity, whereas the parameter $a$ is of the order of $\varepsilon$, a small quantity by virtue of the definition of a composite material. According to (1.4), the parameter $b_{12}$ may here take large negative values at positive values of $b_{33}$ comparable with unity. This analysis makes it possible to establish the mechanical significance of condition (2.4): It excludes shells reinforced with two oblique families of fibers and possessing (together with low matrix stiffness) higher compliance in tension and compression than in shear.

Obviously,

$$q^*(\varphi,\psi) = \min_{m,n} Q \leqslant \min_{m=1,n} Q \equiv q_*(\varphi,\psi),$$

where for values of the parameters $\varphi$, $\psi$ satisfying inequality (2.4) we get the equal sign. Hence it follows that if the values $\geq = \varphi_+$, $\psi = \psi_+$ realizing the maximum

$$q_+ = \max_{\varphi,\psi} q_* = \max_{\varphi,\psi}\min_{m=1,n} Q \tag{2.5}$$

satisfy condition (2.4), then these values realize the maximum (2.3), since

$$q^*(\varphi_+,\psi_+) = q_*(\varphi_+,\psi_+) \geqslant q_*(\varphi,\psi) \geqslant q^*(\varphi,\psi),$$

i.e.,

$$q^*(\varphi_+,\psi_+) \geqslant q^*(\varphi,\psi)$$

(for any $\varphi$, $\psi$ from $\Omega$).

Below we consider the auxiliary problem of calculating the extremum (2.5) and the values $\varphi = \varphi_+$, $\psi = \psi_+$, $n = n_+$ that realize it. This problem is solved in two successive stages. In the first stage we distinguish a subregion of the region $\Omega$ for points in which the order of the quantity $q_*$ is greatest and estimate the order of the minimizing values of n corresponding to these points. This enables us to replace the exact problem (2.5) by a certain simplified problem. In the second stage we find explicit solutions of the simplified problem in certain regions of variation of the geometric parameters t, r. For these solutions condition (2.4) proves to be satisfied, so that they are asymptotically exact solutions of the optimization problem posed.

We temporarily impose on the region of variation of the parameters $\varphi$, $\psi$ the requirement

$$a_{22} = O(1), \tag{2.6}$$

which excludes from the region $\Omega$ the neighborhood of the point $\varphi = 1$, $\psi = 0$. We relate the order of the quantities r, n, Q (at m = 1) to the small parameter t:

$$r = O(t^{\alpha}),\ \alpha \geqslant 0,\ n^2 = O(t^{-\beta}),\ \beta \geqslant 0,\ Q|_{m=1} = O(t^{\gamma})$$

($\alpha$, $\beta$, and $\gamma$ are the order exponents). The first of these equations represents a comparative estimate of two independent geometric parameters r and t, whose values are known by condition. Accordingly the value of the exponent $\alpha$ (the condition $\alpha \geq 0$ excludes very short shells) is also known. The other exponents, however, are not known and are determined as a result of an asymptotic analysis of the solution of the optimization problem formulated.

We denote the exponent of the order of the first term in expression (1.6) (at m = 1) by $\gamma_1$ and that of the second by $\gamma_2$. The exponent $\gamma$ is expressed in terms of $\gamma_1$ and $\gamma_2$ as follows:

$$\gamma = \min(\gamma_1, \gamma_2).$$

This exponent is treated as a function of the parameters $\beta$, $\varphi$, and $\psi$ (the value of the parameter $\alpha$ is assumed to be fixed). Let

$$q_* = O(t^{\gamma_*}),\quad q_+ = O(t^{\gamma_+});$$

then in accordance with (2.5),

$$\gamma_+ = \min_{\varphi,\psi} \gamma_*,\ \gamma_* = \max_{\beta} \gamma = \max_{\beta} [\min(\gamma_1, \gamma_2)], \tag{2.7}$$

by using relations (2.6), (1.4), and (2.2) it can be shown that

$$Q_1|_{m=1} = O(1),\ Q_2 = O(t^{-\tau}),\ \tau \geqslant 0\ (Q_2 > 1/(1+\varepsilon))$$

and, hence,

$$\gamma_1 = 1 - \beta,\ \gamma_2 = 4\alpha + 3\beta - 3 + \tau(\beta, \varphi, \psi). \tag{2.8}$$

Since $\gamma_2$ is an increasing and $\gamma_1$ a decreasing quantity with respect to $\beta$, we obtain [$\beta_*$ is the value of the parameter $\beta$ realizing the maximum (2.7)]

$$\gamma_* = \begin{cases} 1, & \text{if}\quad \gamma_1(0) \leqslant \gamma_2(0), \\ 1 - \beta_0, & \text{if}\quad \gamma_1(0) > \gamma_2(0), \end{cases}\quad \beta_* = \begin{cases} 0, & \text{if}\quad \gamma_1(0) \leqslant \gamma_2(0), \\ \beta_0, & \text{if}\quad \gamma_1(0) > \gamma_2(0), \end{cases} \tag{2.9}$$

where $\beta_0$ is the solution of the equation

$$\gamma_1(\beta) = \gamma_2(\beta). \tag{2.10}$$

The condition $\gamma_1(0) > \gamma_2(0)$, which in expanded form may be written

$$\alpha + \tau(0, \varphi, \psi)/4 < 1,$$

ensures the existence of a solution of Eq. (2.10). From (2.8)-(2.10) there follows the representation

$$\gamma_*(\varphi, \psi) = \begin{cases} 1, & \text{if}\quad \alpha + \tau(0, \varphi, \psi)/4 \geqslant 1, \\ \alpha + \tau(\beta_0, \varphi, \psi), & \text{if}\quad \alpha + \tau(0, \varphi, \psi)/4 \leqslant 1, \end{cases}$$
$$\beta_*(\varphi, \psi) = \begin{cases} 0, & \text{if}\quad \alpha + \tau(0, \varphi, \psi)/4 \geqslant 1, \\ 1 - \alpha - \tau(\beta_0, \varphi, \psi)/4, & \text{if}\quad \alpha + \tau(0, \varphi, \psi)/4 \leqslant 1, \end{cases}$$

from which we obtain (since $\tau \geq 0$)

$$\gamma_+ = \begin{cases} 1, & \text{if}\quad \alpha \geqslant 1, \\ \alpha, & \text{if}\quad \alpha \leqslant 1, \end{cases}\quad \beta_+ = \begin{cases} 0, & \text{if}\quad \alpha \geqslant 1, \\ 1 - \alpha, & \text{if}\quad \alpha \leqslant 1 \end{cases} \tag{2.11}$$

($\beta_+$ is the value of $\beta$ corresponding to $\gamma_+$). These values are realized for parameters $\varphi$ and $\psi$ satisfying the conditions

$$a_{22}=O(1),\ \tau(1-\alpha,\ \varphi,\ \psi)=0,\quad \text{if}\quad 0\leqslant\alpha\leqslant 1,\ a_{22}=O(1),\quad \text{if}\quad \alpha>1,$$

which are equivalent to the following:

$$\varepsilon+1-\varphi-\psi=O(1),\ \varepsilon\nu_0+2\psi\geqslant O(t^{1+\alpha}),\qquad (2.12)$$

$$a=O(1),\quad \text{if}\quad 0\leqslant\alpha\leqslant 1;$$

$$\varepsilon+1-\varphi-\psi=O(1),\quad \text{if}\quad \alpha>1.\qquad (2.13)$$

From this analysis it follows that if $0\leq\alpha\leq 1$, then in calculating the quantity $q_+$ from Eq. (2.5) both terms of expression (1.6) are important. If, however, $\alpha>1$, then the first term of the expression is decisive, since at $\beta=\beta_+$

$$\gamma_1=1,\ \gamma_2=1+4(\alpha-1)+\tau.\qquad (2.14)$$

This completes our investigation of the order of $q_*$ in the part of region $\Omega$ in which condition (2.6) is satisfied. As a result of a similar investigation, too lengthy to reproduce in detail, it was established that in the rest of the region $\Omega$ (neighborhood of the point $\varphi=1$, $\psi=0$) the order of $q_*$ was less than in the subregions defined by the conditions (2.12), (2.13). As follows from (2.11), in these subregions

$$r^2n_*^{-2}=O(t^\delta),\quad \delta=\begin{cases}1+\alpha, & \text{if}\quad 0\leqslant\alpha\leqslant 1,\\ 2\alpha, & \text{if}\quad \alpha>1\end{cases}\qquad (2.15)$$

($n_*$ is the value of n that minimizes Q at m = 1), so that expression (1.6) can be reduced (at m = 1) to the simpler form

$$Q=t(n^2-1)a_{22}+\frac{r^4n^{-6}t^{-3}}{(1-n^{-2})b_{22}+2b_{33}r^2n^{-2}}\qquad (2.16)$$

From this simplified expression $q_*$ may be calculated with an error of the order of $t^\delta$ [$\delta$ is determined from (2.15)] in subregions (2.12), (2.13) and with a greater error in the rest of $\Omega$. However, since, in this case the maximum value of $q_*$ is realized in precisely these subregions, it is calculated with an error of the order of $t^{\delta}_*$, even if Eq. (2.16) is used over the entire region $\Omega$.

We will show that where the value of the parameter $\alpha$ does not lie close to unity, expression (2.16) can be simplified, so that with a certain degree of accuracy auxiliary problem (2.5) may be solved analytically.

Let the value of $\alpha$ lie on the interval $0\leq\alpha\leq 1$. By means of the straight line

$$\psi=\psi_0\equiv\begin{cases}t^{\alpha+1/2}-(1-\nu_0)\,\varepsilon/2, & \text{if}\quad t^{\alpha+1/2}\geqslant(1-\nu_0)\,\varepsilon/2,\\ 0, & \text{if}\quad t^{\alpha+1/2}\leqslant(1-\nu_0)\,\varepsilon/2,\end{cases}\qquad (2.17)$$

we divide the region $\Omega$ into two subregions

$$\psi_0\leqslant\psi\leqslant\varphi(1-\varphi),\ 0\leqslant\varphi\leqslant 1;\qquad (2.18)$$

$$0\leqslant\psi\leqslant\min\,[\psi_0,\ \varphi(1-\varphi)],\ 0\leqslant\varphi\leqslant 1\qquad (2.19)$$

(at $\psi_0=0$ there is only one subregion, coinciding with the entire region $\Omega$). In subregion (2.18)

$$b_{33}r^2n_*^{-2}\leqslant O(t^{1/2}),$$

so that, by admitting an error of the order of $t^{1/2}$, we can neglect this term by comparison with $b_{22}$. Since in subregion (2.19) $\psi\leq t^{\alpha+1/2}$, and, in accordance with (2.12), the value $\varphi=\varphi_+$ realizing extremum (2.5) is such that

$$\varphi_+=O(1),\ 1-\varphi_+=O(1),$$

we may assume that

$$a_{22}\approx a_{22}^0=1-\varphi+\varepsilon,\quad b_{22}\approx b_{22}^0=1/(\varphi+\varepsilon).$$

As a result, we arrive at the following simplified expressions:

$$Q=t(n^2-1)a_{22}+\frac{r^4n^{-6}t^{-3}}{(1-n^{-2})b_{22}}\quad \text{[in subregion (2.18)]},$$

$$Q=t(n^2-1)a_{22}^0+\frac{r^4n^{-6}t^{-3}}{(1-n^{-2})b_{22}^0+2b_{33}r^2n^{-2}}\quad \text{[in subregion (2.19)]}.\qquad (2.20)$$

The first of these expressions is a decreasing and the second an increasing function of $\psi$, since

$$\partial a_{22}/\partial\psi < 0,\ \partial b_{22}/\partial\psi > 0,\ \partial b_{33}/\partial\psi < 0,$$

and hence the quantity $q_*$ reaches its greatest value of the line $\psi = \psi_0$; i.e., $\psi_+ = \psi_0$. Accordingly, in calculating extremum (2.5), with an error of the order of $t^{1/2}$ instead of (2.16) we can use the expression

$$Q = t\,(n^2 - 1)\,a_{22}^0 + \frac{r^4 n^{-6} t^{-3}}{(1 - n^{-2})\,b_{22}^0}. \tag{2.21}$$

If the value of the parameter $\varphi$ is not very close to unity, then it is possible to neglect $n_*^{-2}$ as compared with unity, since for the values of the parameters $\varphi$, $\psi$ realizing maximum (2.5), in terms of order, by virtue of (2.11)

$$n_*^{-2} = O\,(t^{1-\alpha}).$$

Accordingly, expression (2.21) reduces to the simpler form

$$Q = t\,n^2\,a_{22}^0 + r^4 n^{-6} t^{-3}/b_{22}^0.$$

The minimum of this quantity with respect to n, and then the maximum of $q_*$ with respect to $\varphi$ are calculated analytically. As a result, for the quantities realizing extremum (2.5), we obtain the following expressions:

$$\begin{gathered} q_+ = r[1 + O(t^{1/2} + t^{1-\alpha} + \varepsilon)],\ 0 \leqslant \alpha < 1, \\ n_+ = \langle \sqrt{t^{-1} r} \rangle,\ \varphi_+ = 1/4,\ \psi_+ = \psi_0 \end{gathered} \tag{2.22}$$

(the angled brackets denote the integral part of the number). The value of $\psi_0$ is given by (2.17).

We note that if $\varepsilon = O(t^\rho)$, then at $1/2 + \alpha \leq \rho \leq 1 + \alpha$ it is possible to assume that $\psi_+ = + = 0$, and result (2.22) is correct to within an error of the order of $t^{1+\alpha-\rho}$. When $\rho > 1 + \alpha$, although the value of $\psi_0$ is close to zero, it cannot be assumed that $\psi_+ = 0$, since the maximum of $q_*$ with respect to $\varphi$, which at $\psi = 0$ is of the order of $t^{\alpha+\delta}$ $(\delta = (\rho - \alpha - 1)/3 > 0)$, is much less than the value of (2.22), which is of the order of $t^\alpha$.

Relations (2.22) give the complete solution of the auxiliary problem on the interval $0 \leq \alpha < 1$. On the interval $\alpha > 1$ it follows from (2.14) that for Q it is possible to take the expression

$$Q = t(n^2 - 1)a_{22},$$

from which we find the complete solution of the auxiliary problem in the form

$$q_+ = 3t\{1 + O[t^{2\alpha} + t^{4(\alpha-1)} + \varepsilon]\},\ n_+ = 2,\ \varphi_+ = \psi_+ = 0. \tag{2.23}$$

For values of $\alpha$ close to unity the auxiliary problem cannot be solved in explicit form. In solving it numerically it is possible to employ simplified expressions (2.16) and, if $\alpha < 1$, the even simpler expressions (2.21).

The values of the parameters $\varphi_+$, $\psi_+$ determined from relations (2.22), (2.23) satisfy condition (2.4). Accordingly solutions (2.22), (2.23) of the auxiliary problem give the solutions of the optimization problem in the regions of variation of the parameters t, r corresponding to the intervals of variation of the parameter $\varphi$ considered. For convenience, we present these solutions expressed in dimensional parameters.

Region A. The parameters t, r satisfy the relation

$$r = O(t^\alpha),\ 0 \leqslant \alpha < 1$$

(shells of "medium" length).

We have the following expressions for the optimal values:

$$p_+ = \sqrt{12}\,\pi\omega\,\frac{R}{L}\left(\frac{H}{\sqrt{12}R}\right)^{5/2}\mathrm{E},\quad n_+ = \left\langle \sqrt{\left(\frac{\sqrt{12}R}{H}\right)^{1/2}\frac{\pi R}{L}} \right\rangle,\quad \varphi_+ = 1/4,\ \psi_+ = \psi_0 \tag{2.24}$$

[$\psi_0$ is found from (2.17)]. The optimal reinforcement structures corresponding to the values $\varphi = \varphi_+$, $\psi = \psi_+$ are determined as a result of solving system of equations (2.1). For the values $\varphi_+ = 1/4$, $\psi_+ = 0$ there is a unique optimal reinforcement structure defined by two families of fibers: The first is directed along the generator and has a volume fraction $\omega_1 = \omega/4$, while the second is directed along the circumference and has a volume fraction $\omega_2 = 3\omega/4$.

Region B. The parameters t, r satisfy the relation

$$r = O(t^\alpha),\ \alpha > 1$$

("long" shells). For the optimal values we have

$$p_+ = 3\sqrt{12}\,\omega\left(\frac{H}{\sqrt{12}R}\right)^3 E,\quad n_+ = 2,\ \varphi_+ = \psi_+ = 0. \qquad (2.25)$$

The values $\varphi_+ = 0$, $\psi_+ = 0$ correspond to a unique optimal reinforcement structure defined by a single circumferential family of fibers with $\omega_2 = \omega$.

The accuracy with which optimal values (2.24), (2.25) are calculated is indicated in (2.22) and (2.23), respectively. For values of the parameter $\alpha$ close to unity relations (2.24), (2.25) give a substantial error and are therefore unsuitable. Above it was recommended that in this region of values of $\alpha$ the auxiliary problem be solved numerically. The fact that in regions A and B the values $\varphi_+$, $\psi_+$ satisfy condition (2.4) suggests that this condition is also satisfied in the region $\alpha \approx 1$, so that the solution of the auxiliary problem is a solution of the optimization problem. Otherwise it is necessary to solve the optimization problem by starting from the complete expression (1.6), minimizing it with respect to m and n, and then maximizing with respect to $\varphi$ and $\psi$, in accordance with (2.3).

If the shells are subjected to uniform normal hydrostatic pressure of intensity p, the results obtained also give the solution of the optimization problem in the regions in question without any increase in error.

Since stability condition (1.5) is formulated in terms of the investigated quantity Q, in the process of solving the optimization problem we also investigated the solution of the stability problem for a hinged orthotropic cylindrical shell subjected to uniform normal lateral or hydrostatic pressure. The following assertions were proved: 1) The value m = 1 (m is the number of longitudinal waves) is critical (minimizes Q) at least in the region of values of the coefficients of orthotropy bounded by condition (2.4); 2) for shells of "medium" length ($0 \le \alpha < 1$) in the region of values of the coefficients of orthotropy formed by the intersection of subregions (2.12), (2.18) it is permissible to simplify the stability condition so as to make possible explicit minimization with respect to the number or circumferential waves n (the corresponding formula for the critical pressure is an analog of the Papkovich formula for isotropic cylindrical shells); 3) for "longer" shells, in this region of values of the coefficients of orthotropy it is permissible to adopt a "semimembrane" formulation of the stability condition based on (2.12); 4) the "semimembrane" formulation admits a generalization of the form (2.06) which, without affecting the accuracy, is valid for shells of "medium" and greater length over the broader region of values of the coefficients of orthotropy defined by conditions (2.12), (2.13). However, whereas the problem of optimization with respect to critical pressure may be completely solved on the basis of stability condition (1.5) simplified in the sense of (2.16), in the stability problem itself such simplification is permissible only in a certain (see above) region of values of the coefficients of orthotropy.

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